A Tour of Categorical Symmetry

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Def . A categorical symmetry is a category acting as a symmetry of a (quantum) field theory.

In this talk:

- What this means and why it's a good thing to study.
- Bulk-boundary correspondence, classification of gapped phases, anomalies, gauging…
- Examples from $c = 1$ CFTs in 1+1d

The simplest notions of symmetry are tied to transformations (translation, rotation, reflection)

$$G \circlearrowright M$$

M symplectic ("phase space")

G acts by symplectomorphisms

Equations of motion given by Hamiltonian flow by $H : M \to \mathbb{R}$

H ("the Hamiltonian" or "energy") must be G-invariant

Noether's Theorem:

If G is a Lie group, in nice circumstances there exists a moment map $Q : M \to \mathbb{R}$ which generates the infinitesimal action by Hamiltonian flow.

Q is "conserved" because

$$\frac{dQ}{dt} = \{H, Q\} = \frac{dH}{ds} = 0$$

The simplest notions of symmetry are tied to transformations (translation, rotation, reflection)

$$G \circlearrowright M$$

Quantum mechanics:

$$M = P(\mathscr{H}) \qquad \rho : G \to PU(\mathscr{H})$$

$$i\hbar \frac{d\psi}{dt} = H\psi \qquad \rho(g)H = H\rho(g)$$

$$\rho(e^{i\theta^a t_a}) = e^{i\theta^a \rho(t_a)} \qquad \frac{d\rho(t_a)}{dt} = [H, \rho(t_a)] = 0$$

In classical field theory, M is typically sections of some bundle over a space X

$$H = \int_{\text{space}} h(\phi) \qquad\qquad S = \int_{\text{spacetime}} L(\phi)$$

$$Q = \int_{\text{space}} q(\phi)$$

Symmetries coming from action on the fibers give rise to charge densities

$$0 = \frac{dQ}{dt} = \int_{\text{space}} \frac{dq}{dt} \Longrightarrow \frac{dq}{dt} = dj$$

We get a strengthening of Noether's theorem

Passing to spacetime $X \times \mathbb{R}$, (q,j) defines a D-1 form, symmetry <=> conservation law <=> $dJ = 0$.

We have a family of “topological operators” $\int_Y J$ which depend only on the topology of Y

This is equivalent to the conservation of J, and we can construct topological operators from discrete symmetries as well.

In quantum field theory, there might be no classical limit, or many classical limits, so it is ambiguous to talk about symmetries of the fields in this sense.

Topological operators are unambiguous and we will take them to be the *definition* of a symmetry.

Eg. S^1 sigma model in D spacetime dimensions, $\phi : X \to \mathbb{R}/\mathbb{Z}$

$$S = R^2 \int_X |d\phi|^2 = R^2 \int_X d\phi \wedge \star d\phi$$

There is an ordinary symmetry acting by rotation $\phi \mapsto \phi + \alpha$. It is generated by $J = \star\, d\phi$.

The corresponding current is conserved by the equations of motion $d \star d\phi\; = 0$.

There is also a 1-form current $d\phi$ which is conserved simply because $d^2 = 0$.

However, this current does not act on any operator $f(\phi)$!

Around non-contractible loops γ, we can have a winding = $\int_\gamma d\phi \in \mathbb{Z}$

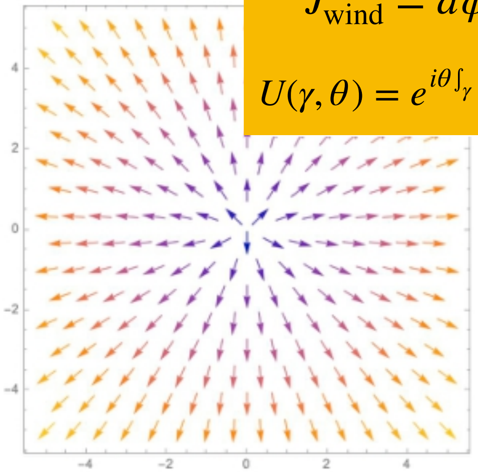
We can define a new kind of observable by removing a codim 2 submanifold Σ from spacetime and imposing winding boundary conditions for the field on $\partial(N\Sigma) \sim \Sigma \times \gamma$.

These operators describe the response to "vortices".

1+1d T-duality: $d\phi = \star\, d\tilde{\phi}$ the vortex is $e^{i\tilde{\phi}}$ in dual variables

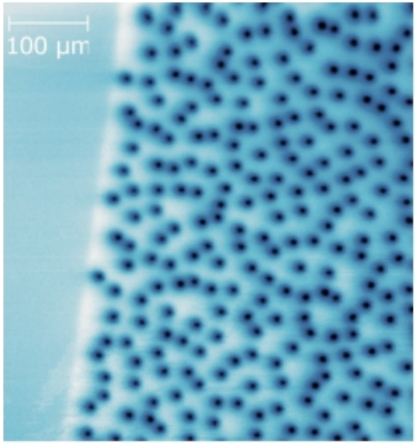
No description of the symmetry is complete without including this “hidden” symmetry.

No reason not to include $d\phi$ among the set of currents for D>2 either.



$$J_{\text{wind}} = d\phi$$

$$U(\gamma, \theta) = e^{i\theta \int_\gamma d\phi}$$



In fact, for D > 2 there is a duality between ϕ and a (D-2)-form gauge field A, such that

$$d\phi = \star\, dA$$

The theory of the gauge field can be written

$$S = R^{-2} \int dA \wedge \star dA$$

The current $d\phi$ generates a global symmetry $A \mapsto A + \lambda$, where λ is a flat (D-2)-form connection.

We consider these as symmetries even though they don't act on local operators.

This one acts on Wilson operators

$$e^{i \int_\Sigma A}$$

How do we describe the set of topological operators? For example, topological lines in 1+1d

Every line carries a label $a \in \mathscr{A}$

This set forms a linear category with morphisms $\mathrm{Hom}(a, b)$ given by topological point operators between lines.

There is a fusion product $a \otimes b \in \mathscr{A}$. In the case of a symmetry group, this is the group multiplication.

Junctions in $\mathrm{Hom}(a \otimes b, c)$ define local fusion rules, so we can consider networks of lines.

There is also the F-symbol or crossing relation

For symmetry groups G, this F-symbol captures the 't Hooft anomaly, $\mathscr{A} = \mathrm{Vec}_G^{\Omega}$, $\Omega \in H^3(G, U(1))$

The data of the fusion product and the F symbol gives $\mathscr{A}$ the structure of a *monoidal category* .

This has a close relation to a notion of the category of 2d QFTs, with morphisms given by topological defects.

In this picture, D spacetime dimensional QFTs form a D-category.

A and its generalizations are given by the endomorphisms of our theory in this category.

Endomorphisms in a D-category form a monoidal (D-1)-category with the product given by composition.

Hard to describe or construct!

What about non-invertible topological operators?

- The non-invertible topological operators will also be preserved by dualities.

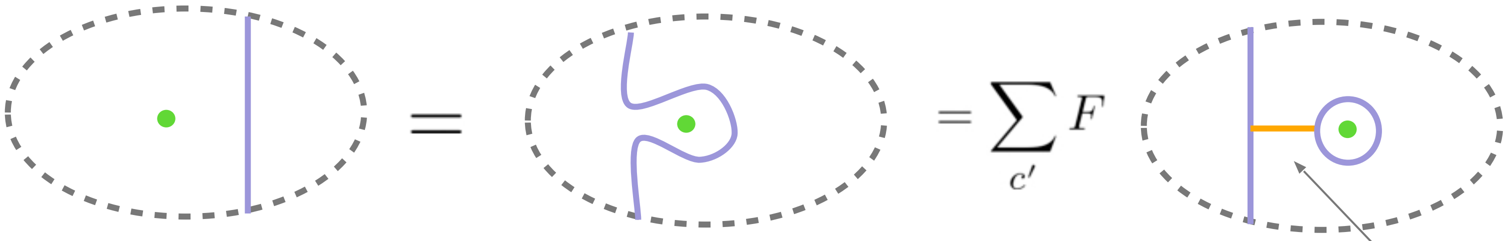
- If we consider renormalization group flows generated by “symmetric” perturbations, all topological operators will remain topological along the flow (although there can be spontaneous symmetry breaking). Symmetric means:

- In fact the whole data of A will be preserved => ‘t Hooft anomaly matching

Eg. The critical Ising model in 1+1d is self-dual under $\mathbb{Z}_2$ gauging, which can be written as:

$$H = -\sum_n X_n X_{n+1} + Z_n \qquad X_n = \tilde{Z}_n \tilde{Z}_{n+1} \cdots \quad Z_n = \tilde{X}_{n-1} \tilde{X}_n$$

- Such a transformation is not an invertible symmetry because it does not preserve the set of local operators. It is associated with a non-invertible topological operator.



Sum over tails $c' \in a \otimes \bar{a}$

- This gives the fusion rule $\sigma \otimes \sigma = 1 \oplus \epsilon$

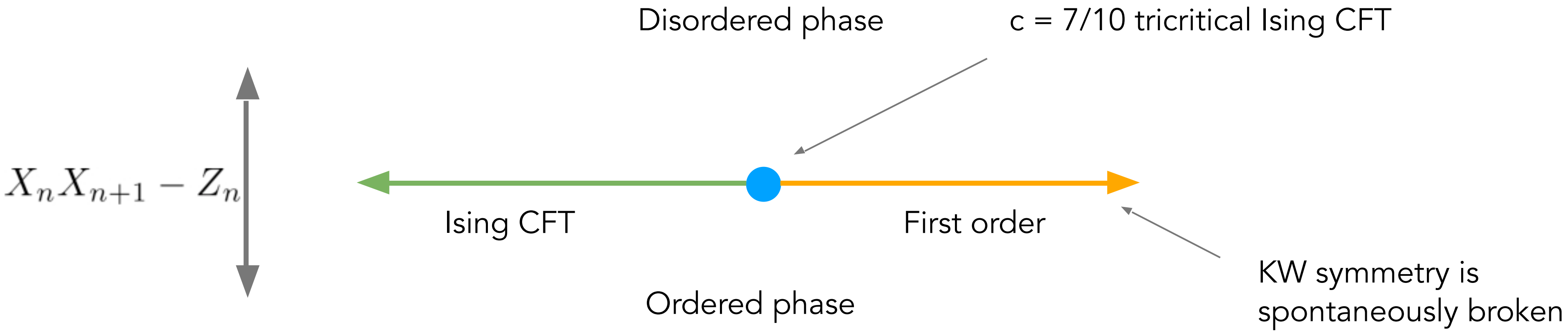
- There are two consistent choices for the F symbol, can determine which one by modularity.

We have seen that the order parameter is not symmetric. Neither is the "energy operator"

$$X_n X_{n+1} - Z_n$$

which tunes between the ordered and disorder phases. Its sign is flipped by KW because these phases are exchanged by the transformation.

There are symmetric irrelevant operators which lead to an interesting phase diagram:



These symmetries naturally occur at the boundary of TQFTs.

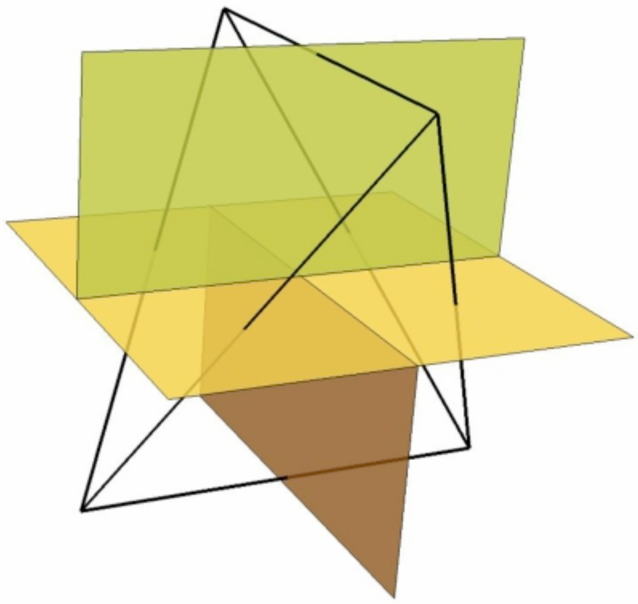
If $\mathscr{A}$ is suitably finite (eg. a fusion category) then we can construct a state-sum TQFT from it via the Turaev-Viro/Levin-Wen construction.

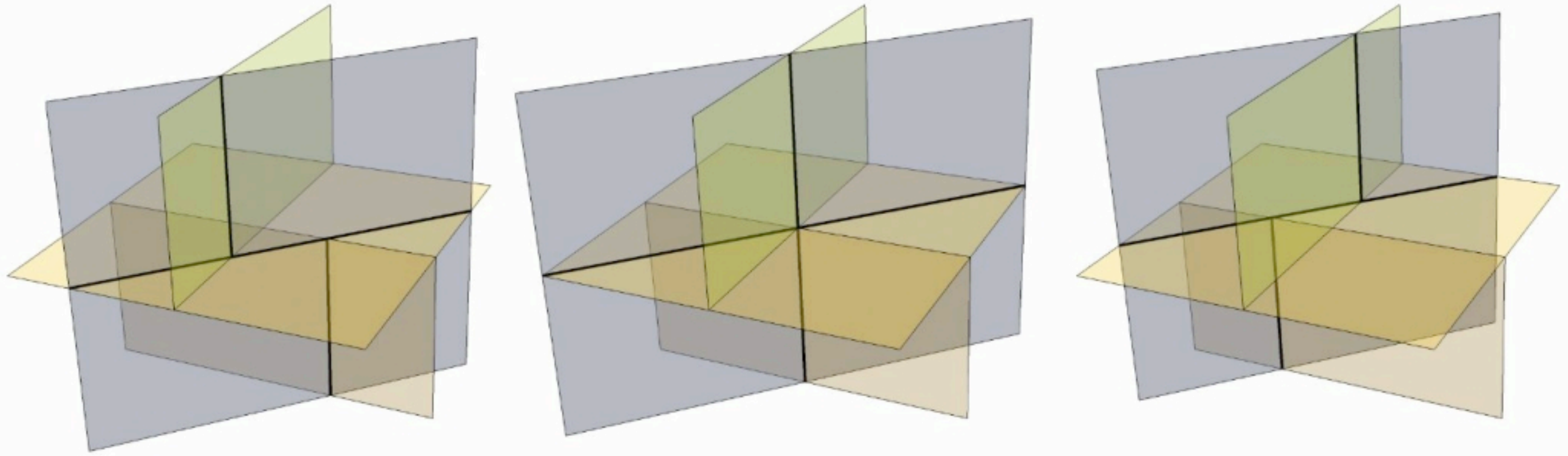
Intuitively, the partition function of this TQFT is a sum over networks with codimension k submanifolds labelled by k-1 morphisms.

More precisely we sum over diagrams in a D-category $\underline{\mathscr{A}}$ (in which $\mathscr{A}$ occurs as endomorphisms of the unique object) which may be embedded in spacetime.

Each 0-dimensional junction gives an F symbol to be evaluated and the weight of the diagram is the product of all these numbers.

The state-sum is well-defined because there are only finitely many topological classes of networks.





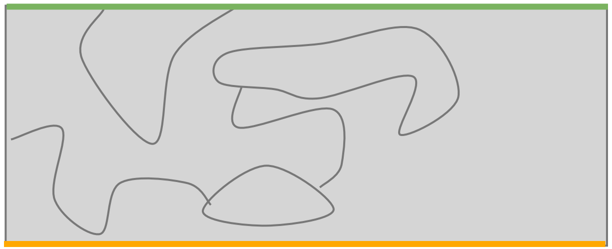
Prop. D-dimensional theories with categorical symmetry $\mathscr{A}$ are in bijection with boundary theories of Turaev-Viro theory defined by $\mathscr{A}$.

Given a theory with categorical symmetry $\mathscr{A}$ we can produce a boundary theory by letting bulk surfaces end on topological operators. The network remains topologically invariant and the state-sum is well-defined.

Conversely, if we have a boundary theory, we can consider a slab with our boundary theory on one side and the Dirichlet boundary condition on the other side where by definition no bulk network can end.

For any $a \in \mathscr{A}$, we can modify the state-sum by imposing other boundary conditions for the network, defining topological operators in the compactified theory.

Boundary theory



Turaev-Viro theory

Dirichlet boundary condition

Eg. $\mathscr{A} = \mathrm{Vec}_{\mathbb{Z}_2}$

The Turaev-Viro state-sum is a sum over $\mathbb{Z}_2$ D-cycles (hypersurfaces).

In Poincare duality, this is equivalently a $\mathbb{Z}_2$ 1-cocycle, ie. a flat $\mathbb{Z}_2$ gauge field A.

Any theory with an anomaly-free $\mathbb{Z}_2$ symmetry can be gauged, defining a good boundary condition for this theory.

The Dirichlet boundary condition is equivalent to taking A = 0 on the boundary.

Forming the slab is thus equivalent to turning off the gauge field, which returns us to the original theory with its global $\mathbb{Z}_2$ symmetry.

It is equivalently an anyon condensate (of "e"s), which we can think of as gauging the "magnetic symmetry".

Eg. $\mathscr{A} = \mathrm{Vec}_{\mathbb{Z}_2}$

The phase diagram of boundaries of this theory matches the phase diagram of $\mathbb{Z}_2$ symmetric phases. Gapped boundaries <=> gapped phases.

There are three anyons in the bulk theory, "e", "m", and their product "em". e and m are bosons and have a mutual braiding, while em is a fermion. Either e or m can become condensed at the boundary.

These correspond to the symmetry breaking and symmetry preserving $\mathbb{Z}_2$ symmetric gapped phases. But which is which?

In the Turaev-Viro construction, we are either summing over e or m lines. These define our global symmetry.

The e condensate corresponds to the symmetry breaking phase, with the order parameter given by the e tunnelling operator through the slab.

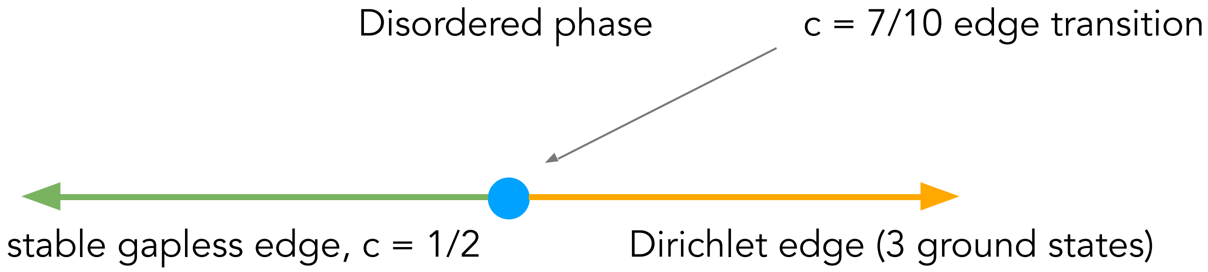
There is a c = ½ boundary phase transition.

Eg. $\mathscr{A} = \mathrm{Ising}^+$, $\sigma \otimes \sigma = 1 \oplus \epsilon$

The Turaev-Viro theory of the Ising category is more interesting - it hosts nonabelian anyons.

By the bulk-boundary correspondence, the boundary theories are equivalent to 1+1d systems with Kramers-Wannier symmetry.

It reproduces the self-dual Ising phase diagram.



The bulk-boundary correspondences gives us a classification of all gapped phases with categorical symmetry $\mathscr{A}$, since we can classify gapped boundaries of Turaev-Viro theory.

These boundaries are classified by module categories of $\mathscr{A}$.

For example, the Dirichlet condition is given by $\mathscr{A}$ acting on itself.

Of special interest are module categories with a single simple object (aka fiber functors). These correspond to SPT-like phases with a unique ground state and an action of $\mathscr{A}$.

Like SPT (symmetry protected topological) phases, these have symmetry-protected edge modes between them. However, there is not necessarily a trivial phase.

Eg. $\mathscr{A} = Rep(D_8)$ has three fiber functors which are related by a triality. There are protected edge modes between any pair.

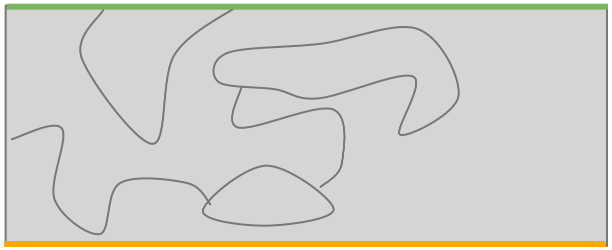
Not every $\mathscr{A}$ admits a fiber functor.

Recall if $\mathscr{A}$ is grouplike it includes the data of the ‘t Hooft anomaly, which forbids a symmetric gapped phase with a unique ground state.

We might say more generally that $\mathscr{A}$ is anomalous if it does not admit a fiber functor.

Conversely, if $\mathscr{A}$ admits a fiber functor, there is a way to gauge it:

Boundary theory



Fiber functor boundary

Different choices of fiber functors are like discrete torsion of orbifolds.

There is a dual categorical symmetry given by the module endomorphisms of the fiber functor, analogous to the magnetic symmetry.

Let's look again at the S^1 sigma model. $$S = R^2 \int_X |d\phi|^2 = R^2 \int_X d\phi \wedge \star d\phi$$

It has continuous symmetry $U(1)_{\text{rotation}} \times U(1)_{\text{winding}}$ as well as a disconnected component generated by these and a reflection symmetry $C : \phi \mapsto -\phi$.

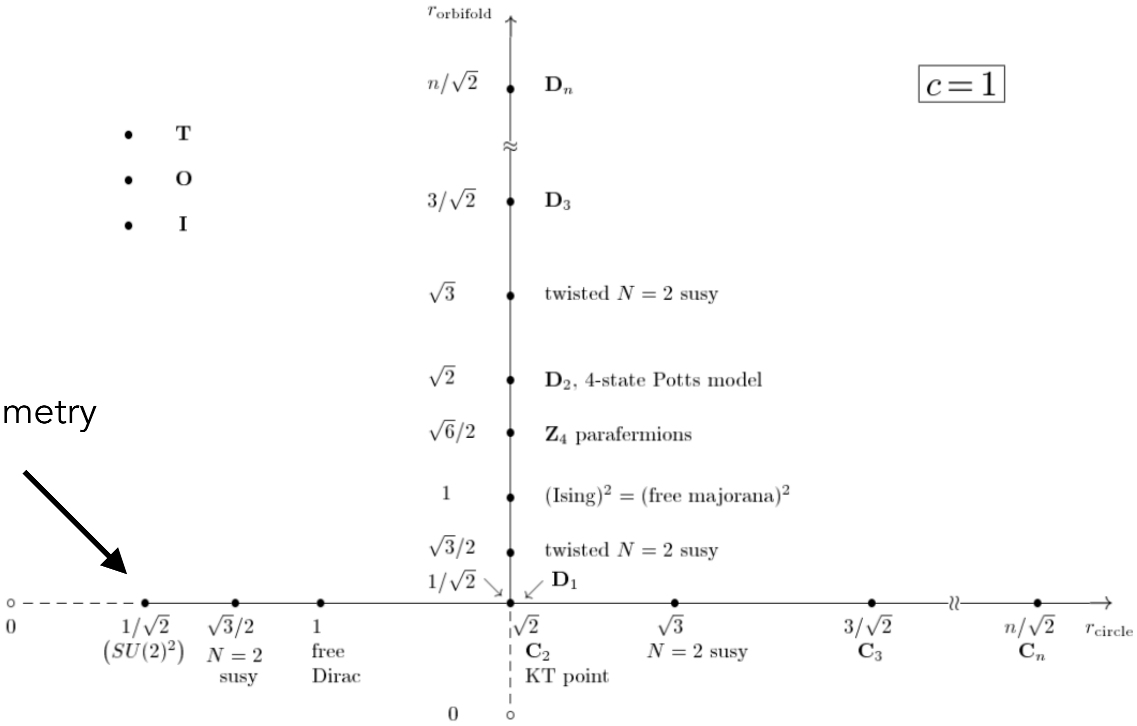
This theory has T-duality, inverting R and exchanging the $U(1)$'s. This is a moduli space $R \geq R_{\text{selfdual}}$.

We can also gauge / orbifold C and produce another family of CFTs with a parameter $\tilde{R} \geq \tilde{R}_{\text{selfdual}}$. These have only a finite symmetry group D_8.

These two families meet at $R = 2R_{\text{selfdual}}$ and $\tilde{R} = \tilde{R}_{\text{selfdual}}$.

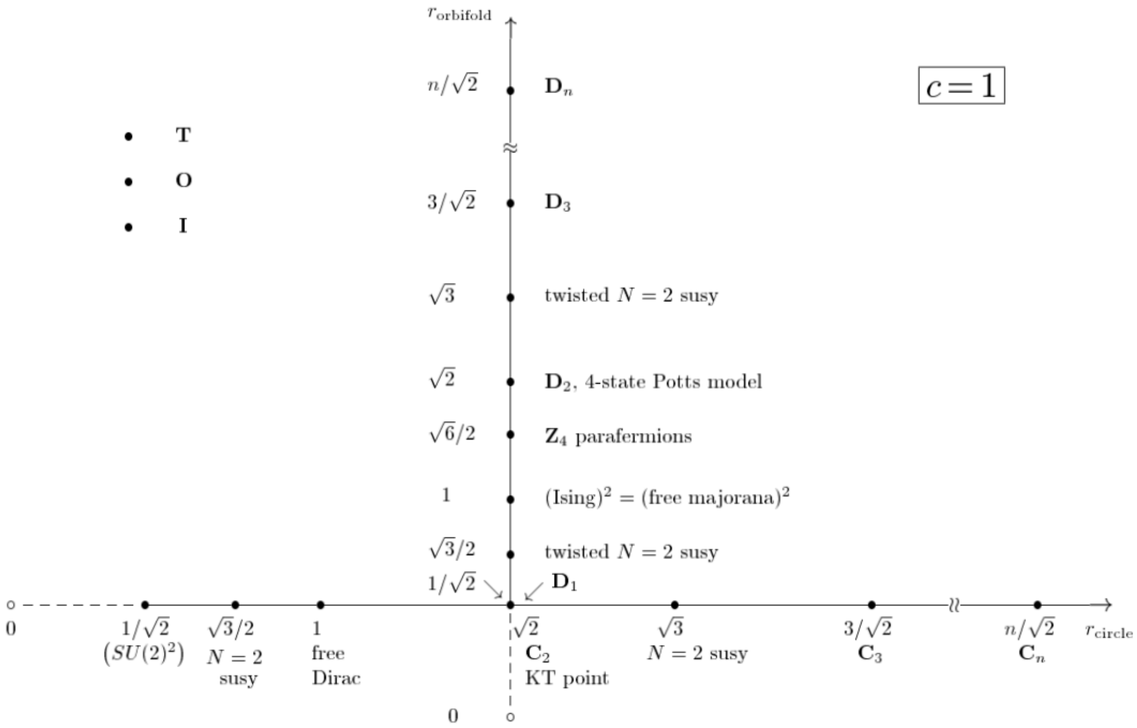
Here is the moduli space (from Ginsparg). There are lots of categorical symmetries hiding!

SO(4) symmetry



Recall that forming the G orbifold of a theory means summing over networks of G topological operators.

Such correlation functions can be modified by inserting Wilson lines $\mathrm{Tr}_R \mathscr{P} e^{i \int_\gamma A}$. These define a $Rep(G)$ categorical symmetry.



Every point on the orbifold branch has $Rep(D_{2n})$ symmetry for every n!

The continuous symmetries of the S^1 compact boson define topological lines, such as the phase slip operator L_α:

$$\lim_{x\to 0^-} \phi(x,t) = \lim_{x\to 0^+} \phi(x,t) + \alpha$$

L_α is not C-invariant. It transforms as $CL_\alpha C = L_{-\alpha}$.

However the sum of lines $L_\alpha + L_{-\alpha}$ is C-invariant, and defines a *continuous family* of non-invertible topological operators on the orbifold branch!

At rational radius, the theory is an orbifold of the self-dual point and get a whole SO(4) family of non-invertible topological lines.

All the categorical symmetries we found, such as the $Rep(D_{2n})$ come about this way. Maybe this is everything?

Any time we have a continuous family of conformal defects the parameter of that family is associated with an exactly marginal operator on the defect.

In the case of topological defects, this marginal operator defines a spin 1, dimension 1 operator, which is not necessarily local but sits at the end of a line

$$l \in L_\alpha \otimes L_{-\alpha}$$

For example the case of the winding symmetry of the S^1 compact boson, this "twist current" may be expressed as

$$j(x) = d\phi(x) e^{i\int_{-\infty}^{x} A}$$

Where A is the $\mathbb{Z}_2^C$ gauge field. It is conserved in the sense that $(d + iA)j = 0$.

Are there such currents outside of finite gauge theories? What are their consequences?

- Topological operators generalize the classical notions of symmetry and are preserved under dualities and symmetric RG flows.
- Topological operators of codimension > 1 and non-invertible ones are beyond the familiar theory of symmetry groups. Instead we must use (monoidal) category theory.
- For finite categories of topological operators, we have a bulk-boundary correspondence: theories with symmetry $\mathscr{A}$ = boundary theories of Turaev-Viro theory constructed from $\mathscr{A}$.
- This lets us classify gapped phases with symmetry $\mathscr{A}$ and define a notion of $\mathscr{A}$ anomaly and $\mathscr{A}$ gauging.
- c = 1 examples reveal that there are continuous families of topological operators. How can we describe them mathematically? Is there a bulk-boundary correspondence?
- We only discussed examples that are related to gauging finite symmetries. In 1+1d there are also non-"group-theoretical" categories such as the Fibonacci anyons, which act as a symmetry of the tricritical Ising model. Are there such non-group-theoretical categories in higher dimensions?